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$\aleph_\omega$ MAY HAVE A STRONG PARTITION RELATION

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ABSTRACT

We prove the consistency, with ZFC + G.C.H., of a strong partition relation of $\aleph_\omega$, assuming the consistency of the existence of infinitely many compact cardinals.

The Erdos–Rado theorem and related partition theorems (see Erdos, Hajnal and Rado [3]) have been very useful. Unfortunately, the really good partition theorems are true only for large cardinals. So a natural question is: what is the best partition theorem which a small cardinal may satisfy? This may be a way to give independence results (and usually $V = L$ will give the negation).

In Shelah [8], answering a question of Erdos and Hajnal [1], [2], we gave such a partition theorem for $\aleph_\omega$ which is consistent with ZFC + G.C.H. We ask there whether a much stronger partition theorem is consistent too. We shall give here a positive answer, but we use a stronger hypothesis (the consistency of ZFC of the existence of $\aleph_0$ compact rather than measurable cardinals).

On similar assertions proved in ZFC, see Erdos, Hajnal, Mate and Rado [4] and Shelah [7].

NOTATION. Natural numbers are denoted by k, l, m, r, ordinals by i, j, α, β, γ, ξ, ζ, η, ν, cardinals by λ, κ, μ, χ. We define $\beth_\alpha(\lambda)$ by induction on α: $\beth_0(\lambda) = \lambda$, and $\beth_\alpha(\lambda) = \Sigma_{\beta<\alpha} 2^{\beth_\beta(\lambda)}$ for $\alpha > 0$. Let $\lambda^{<\mu} = \Sigma_{\kappa<\mu} \lambda^\kappa$.

If $<$ orders A, $B \subseteq A$, $C \subseteq A$, $a \in A$ *then* $B < a$ means $(\forall x \in B) x < a$, $B < C$ means $(\forall x \in B)$ $(\forall y \in C)$ $(x < y)$, etc.

Let $[A]^\kappa = \{B : B \subseteq A, |B| = \kappa\}$, $[A]^{<\kappa} = \{B : B \subseteq A, |B| < \kappa\}$.

We define $\kappa^{+\alpha}$ for an infinite cardinal κ and an ordinal α: if $\kappa = \aleph_\beta$ then $k^{+\alpha} = \aleph_{\beta+\alpha}$.

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We define: $\lambda \to (\mu)^n_\chi$ means that for any n-place function F from λ to χ, there is $B \in [\lambda]^\mu$, such that F has a constant value on all increasing n-tuples from B.

1. Definition. $\langle \lambda_\xi : \xi < \theta \rangle$ has a $\langle \kappa(\xi) : \xi < \theta \rangle$-canonical form for $\Gamma = \{\bar r(i)^{\ell(i)}_{\chi(i)} : i < \alpha\}$ [where $\chi(i)$ is a non-zero cardinal, and $\bar r(i) = \langle n_1(i); \cdots; n_k(i) \rangle$, $n_m(i) \geqq 0$ and $\ell(i)$ are natural numbers, and for each $\bar r = \langle n_1; \cdots; n_k \rangle$ we denote $n(\bar r) = \sum_{i=1}^k n_i$, $k(\bar r) = k$, $n_m(\bar r) = n_m$] *if* for every set A_ξ $(\xi < \theta)$, $|A_\xi| = \lambda_\xi$ (and $<$ well orders $\bigcup_{\xi<\theta} A_\xi$, $A_\xi < A_\eta$ for $\xi < \eta$) and functions f_i $(i < \alpha)$, f_i an $n(\bar r(i))$-place function from $\bigcup_\xi A_\xi$ to $\chi(i)$ *there are* $B_\xi \subseteq A_\xi$, $|B_\xi| = \kappa(\xi)$ such that for every i, f_i is $\bar r(u)^{\ell(i)}$-canonical on $\langle B_\xi : \xi < \theta \rangle$. This means that when $\xi_1 < \cdots < \xi_{k(\bar r(i))} < \theta$,

$$a_1 < \cdots < a_{n_1(\bar r(i))} \in B_{\xi_1}, \quad a_{n_1(\bar r(i))+1} < \cdots < a_{n_1(\bar r(i))+n_2(\bar r(i))} \in B_{\xi_2}, \quad \text{etc.},$$

then $f_i(a_1, \cdots, a_{n(\bar r(i))})$ depends on $\xi_1, \cdots, \xi_k, a_1, \cdots, a_{n(\bar r(i))-\ell(i)}$ only (and not on $a_{n(\bar r(i))-\ell+1}, \cdots, a_{n(\bar r(i))}$).

2. Main Theorem. *Assume* ZFC + G.C.H. *is consistent with the existence of infinitely many compact cardinals (we use much less).*

Then ZFC + G.C.H. *is consistent with*:

$$\langle \aleph_{k_1(n)} : n < \omega \rangle \text{ has } \langle \aleph_{k_2(n)} : n < \omega \rangle\text{-canonical forms for}$$

$$\Gamma = \{\langle n, n+1, \cdots, m \rangle^{n+(n+1)+\cdots+m}_{\aleph_{k_2(n)-1}} : n \leqq m < \omega\}$$

$$\text{where} \quad k_1(n) = (n+5)n/2 + n + 1, \quad k_2(n) = (n+5)n/2 + 1.$$

The rest of the paper is dedicated to a proof, via forcing, starting with a model V such that:

3. Hypothesis. G.C.H. holds and there are compact cardinals $\aleph_0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots <$.

On forcing see e.g. Jech [5]. The proof proceeds via various claims and definitions.

4. Definitions. Let $D_n(\lambda, \mu, \chi)$ be the following filter:

(a) It is a filter over Inc(λ, μ) which is the set of increasing sequences of length μ of ordinals $< \lambda$ (if the universe V is not self-evident, we write Inc$(\lambda, \mu)^V$).

(b) The filter is generated by the set of generators, where a generator is

$\mathrm{Ge}(F) = \mathrm{Ge}_n(F; \lambda, \mu, \chi)$

$$= \{\bar{a} \in \mathrm{Inc}(\lambda, \mu): \text{for some } \alpha < \chi \text{ for any } i(0) < \cdots < i(n-1) < \mu,$$

$$F(a_{i(0)}, \cdots, a_{i(n-1)}) = \alpha\},$$

where F is any n-place function from λ to χ.

5. Claim. (1) *If $\chi = \chi^{<\kappa}$ (which holds always for $\kappa = \aleph_0$) then the intersection of $<\kappa$ generators of $D_n(\lambda, \mu, \chi)$ is a generator: hence the filter $D_n(\lambda; \mu, \chi)$ is κ-complete.*

(2) *If $\lambda \to (\mu)^n_\chi$ (the usual partition relation) then $D_n(\lambda, \mu, \chi)$ is a proper filter, i.e., the empty set does not belong to it.*

Proof. Trivial.

6. Notation. Let E_n be a normal ultrafilter over κ_n (exists because as κ_n is compact, it is a measurable cardinal). Let $I_n = \mathrm{Inc}(\kappa_n^{+(n+1)}, \kappa_n^{+1})$ and $J_n = \kappa_n \times I_n$. Note that $D_{n+1}(\kappa_n^{+(n+1)}, \kappa_n^{+1}, \kappa_n)$ is a κ_n-complete (proper) filter (as $\kappa_n^{<\kappa_n} = \kappa_n$, because κ_n is compact, hence strongly inaccessible; and as G.C.H. holds, $D_{n+1}(\kappa_n^{+(n+1)}, \kappa_n^{+1}, \kappa_n)$ is a proper filter). So as κ_n is compact there is a κ_n-complete ultrafilter D_n^* over I_n extending $D_{n+1}(\kappa_n^{+(n+1)}, \kappa_n^{+1}, \kappa_n)$. So

$$F_n = E_n \times D_n^* = \{A \subseteq J_n = \kappa_n \times I_n : \{i < \kappa_n : \{t \in I_n : \langle i, t\rangle \in A\} \in D_n^*\} \text{ is in } E_n\}.$$

We call $f: J_n \to \kappa_n$ regressive if $f(\alpha, t)[\alpha < \kappa, t \in I_n$; more formally $f(\langle \alpha, t\rangle)]$ is an ordinal $< \alpha$. We call it regressive on A if $f(\alpha, t) < \alpha$ for $\langle \alpha, t\rangle \in A$; and almost regressive if it is regressive on some $A \in F_n$. We define, when f is constant, constant on A and almost constant, similarly.

7. Claim. *Every almost regressive function $f: J_n \to \kappa$ is almost constant.*

Proof. Let f be regressive on $B \in F_n$. Let $B_\alpha = \{t \in I_n : \langle \alpha, t\rangle \in B\}$, so for some $B' \subseteq \kappa$, $B' \in E_n$ and $B_\alpha \in D_n^*$ for $\alpha \in B'$.

For each $\alpha \in B'$, $\{A^\alpha_\beta : \beta < \alpha\}$ where $A^\alpha_\beta = \{t \in I_n : f(\alpha, t) = \beta\}$ is a partition of B_α to $|\alpha| < \kappa$ parts. As D_n^* is κ-complete, $B_\alpha \in D_n^*$, for some $\beta = h(\alpha) < \alpha$, $A_{h(\alpha)} \in D_n^*$. So h is a regressive function on B'. Hence as $B' \in E_n$ and E_n is normal, there is $\gamma < \kappa$ such that $\{\alpha : h(\alpha) = \gamma\} \in E_n$. Trivially

$$\{\langle \alpha, t\rangle : f(\alpha, t) = \gamma\} \in E_n \times D_n^* = F_n$$

and of course f is constant on this set.

8. THE FORCING. Let P_n be the Levi collapse of κ_{n+1} to κ_n^{+n+3}; i.e., P_n collapse every λ, $\kappa_n^{+n+1} < \lambda < \kappa_{n+1}$ to κ_n^{+n+2}, and each condition consists of κ_n^{+n+1} atomic conditions of the form $\underset{\sim}{H}{}^n_\lambda(\alpha) = \beta$ (λ as above, $\alpha < \kappa_n^{+n+2}$, $\beta < \lambda$) (see e.g. [5]). The order is inclusion. Let

$$p \restriction \xi = \{\text{“}\underset{\sim}{H}{}^n_\lambda(\alpha) = \beta\text{”} : \underset{\sim}{H}{}^n_\lambda(\alpha) = \beta \text{ belong to } p, \lambda < \xi\}$$

and $\qquad \lambda(p) = \mathrm{Sup}\{\lambda : \text{for some } \alpha, \beta, \underset{\sim}{H}{}^n_\lambda(\alpha) = \beta \text{ belong to } p\}.$

Let $P = \Pi_{n<\omega} P_n$. Let $G \subseteq P$ be generic, $G_n = G \cap P_n$. Let $\phi_n \in P_n$ be the empty condition (so we stipulate $n \neq m$, $\phi_n \neq \phi_m$). We identify $\langle p_0, \cdots, p_{n-1}\rangle \in \Pi_{\ell<n} P_\ell$ with $\langle p_0, \cdots, p_{n-1}, \phi_n, \phi_{n+1}, \cdots\rangle$ and $p \in P_n$ with $\langle \phi_0, \cdots, \phi_{n-1}, p, \phi_{n+1}, \phi_{n+2}, \cdots\rangle$.

As is well known the first ω cardinals in $V[G]$ are $\aleph_0 = \kappa_0$, κ_0^{+1}, κ_0^{+2}, κ_1, κ_1^{+1}, κ_1^{+2}, κ_1^{+3}, κ_2, κ_2^{+1}, κ_2^{+2}, κ_2^{+3}, κ_2^{+4}, $\kappa_3, \cdots, \kappa_n, \kappa_n^{+1}, \cdots, \kappa_n^{+n+1}, \kappa_n^{+n+2}, \kappa_{n+1}, \cdots$. Also $V[G]$ satisfies G.C.H.

Let f be (in $V[G]$) a function from increasing finite sequences from $\aleph_\omega$ to $\aleph_\omega$, such that for $\alpha_0 < \cdots < \alpha_k < \kappa_n^{+n+1}$, $f(\alpha_0, \cdots, \alpha_k) < \kappa_n$ and w.l.o.g. from the value of f for $\langle \alpha_0, \cdots, \alpha_k\rangle$ we can compute its value on any increasing subsequence starting with α_0.

We have to prove that there are sets S_n $(n > 0)$, $S_n \subseteq \kappa_n^{+n+1}$, $|S_n| = \kappa_n^{+1}$, $S_n \cap \kappa_n = \emptyset$, and for every increasing sequence $\alpha_0 < \cdots < \alpha_{k-1}$ of members of $\bigcup_n S_n$, $|S_n \cap \{\alpha_0, \cdots\}|$ is $n+1$ for $n_0 \leqq n \leqq n_1$, and zero otherwise, that $f(\alpha_0, \cdots, \alpha_{k-1})$ depend only on k and the truth values of “$\alpha_\ell \in S_n$”. Moreover, this is sufficient for proving the theorem.

So let $\underset{\sim}{f}$ be a P-name of f, and $p = \langle p_n : n < \omega\rangle \in P$. We shall find p', $p \leqq p' \in P$, and S_n $p' \Vdash_P$ “S_n $(n < \omega)$ are as required”. This clearly suffices.

9. CLAIM. *If* $A \in F_{n+1}$, $p_{\langle\alpha,t\rangle} \in P_n$ *for every* $\langle \alpha, t\rangle \in A$ *then there is* $B \subseteq A$, $B \in F_{n+1}$ *and* $q \in P_n$ *such that*:

$(*)\qquad$ *for any* $\langle \alpha, t\rangle \in B$, $\quad p_{\langle\alpha,t\rangle} \restriction \alpha = q$,

hence

$(**)\qquad$ *for any* $r, q \leqq r \in P_n$, $\quad$ *if* $\lambda(r) < \alpha$, $\quad \langle \alpha, t\rangle \in B$

then $p_{\langle\alpha,t\rangle}$, r *are compatible.*

PROOF. It is easy to prove $(*)$ by the normality of F_n, and $(**)$ follows easily by the definition of P_n.

10. PROOF OF THE THEOREM. We continue 8.

First, as each P_ℓ is $\kappa_\ell^{+(\ell+2)}$-complete, we can find $\bar{p}_0 = \langle p_0^0, p_1^0, \cdots \rangle$, $\bar{p} \leqq \bar{p}_0$, such that for each n:

(0) $\bar{p}_0 \Vdash_P$ "$\underset{\sim}{f} \restriction \kappa_n^{+(n+1)}$ is determined by forcing with $\Pi_{\ell<n} P_\ell$". So for some $\Pi_{\ell<n} P_\ell$-name $\underset{\sim}{f}_n$, $\bar{p}_0 \Vdash$ "$\underset{\sim}{f} \restriction \kappa_n^{+(n+1)} = \underset{\sim}{f}_n$".

Now we define by induction on k, a condition $\bar{p}_k = \langle p_0^k, p_1^k, \cdots \rangle$, sets $A_\ell^k \in F_\ell$ ($\ell < \omega$) and conditions $q_{\langle \alpha, t \rangle}^k \in P_\ell$ ($\langle \alpha, t \rangle \in A_\ell^k$, $\ell < \omega$) such that:

(1) $p_\ell^k \leqq p_\ell^{k+1}$(in P_ℓ), $A_\ell^{k+1} \subseteq A_\ell^k$, $\langle \alpha, t \rangle \in A_{\ell+1}^0 \rightarrow \kappa_\ell < \alpha$;

(2) $q_{\langle \alpha, t \rangle}^k \leqq q_{\langle \alpha, t \rangle}^{k+1}$ for $\langle \alpha, t \rangle \in A_\ell^{k+1}$;

(3) $p_\ell^k \leqq q_{\langle \alpha, t \rangle}^k$, moreover $p_\ell^k = q_{\langle \alpha, t \rangle}^k \restriction \alpha$ (for $\langle \alpha, t \rangle \in A_\ell^k$);

(4) for any n, k for some $\Pi_{\ell<n} P_\ell$-name $\underset{\sim}{f}_n^k$ for any $\langle \alpha_{n+1}, t_{n+1} \rangle \in A_{n+1}^k$, $\langle \alpha_{n+2}, t_{n+2} \rangle \in A_{n+2}^k, \cdots, \langle \alpha_{n+k}, t_{n+k} \rangle \in A_{n+k}^k$ and increasing sequences $\bar{\beta}_{n+\ell}$ from $t_{n+\ell}$ of length $n + \ell + 1$ for $\ell = 1, \cdots, k$,

$$\bar{p}^k \cup \bigcup_{\ell=1}^{k} q_{\langle \alpha_{n+\ell}, t_{n+\ell} \rangle} \Vdash_P \text{"for any increasing sequence } \bar{\gamma} \text{ from } \kappa_n^{+(n+1)}$$

$$\underset{\sim}{f}(\bar{\gamma}, \bar{\beta}_{n+1}, \cdots, \bar{\beta}_{n+k}) = \underset{\sim}{f}_n^k(\bar{\gamma})\text{"}$$

(note that $\bar{p}^k \cup \bigcup_{\ell=1}^k q_{\langle \alpha_{n+1}, t_{n+1} \rangle} = \langle p_0^k, \cdots, p_{n-1}^k, p_n^k, p_{n+1}^k \cup q_{\langle \alpha_{n+1}, t_{n+1} \rangle}, \cdots, p_{n+k}^k \cup q_{\langle \alpha_{n+k}, t_{n+k} \rangle}, p_{n+k+1}^k, \cdots \rangle$).

For $k = 0$. Let $A_n^k = \{\langle \alpha, t \rangle \in J_n : \bigcup_{\ell<n} \kappa_\ell < \alpha < \kappa_n\}$,

$$q_{\langle \alpha, t \rangle}^k = p_n^0 \quad \text{for } \langle \alpha, t \rangle \in A_n^k.$$

For $k+1$. Let $n < \omega$, remember $\underset{\sim}{f}_{n+1}^k$ is a $\Pi_{\ell<n} P_\ell$-name of a function with domain the increasing finite sequences from $\kappa_{n+1}^{+(n+2)}$ and range $\subseteq \kappa_n^{+(n+2)}$ (except on the empty sequence, which is immaterial). Remember that G.C.H. holds, each κ_n is regular and $\Pi_{\ell<(n+1)} P_\ell$ satisfies the κ_{n+1}-chain condition.

So for each sequence $\bar{\alpha} = \langle \alpha_0, \cdots, \alpha_n \rangle$, $\alpha_0 < \cdots < \alpha_{n+1} < \kappa_{n+1}^{+(n+2)}$ there is a set $\{\langle r_i^{\bar{\alpha}}, \gamma_i^{\bar{\alpha}} \rangle : i < i(\bar{\alpha})\}$, $\{r_i^{\bar{\alpha}} : i < i(\bar{\alpha})\}$ a maximal antichain of $\Pi_{\ell \leqq n} P_\ell$, $\gamma_i^{\bar{\alpha}} < \kappa_n^{+(n+2)}$ and $r_i^{\bar{\alpha}} \Vdash$ "$\underset{\sim}{f}_{n+1}^k(\bar{\alpha}) = \gamma_i^{\bar{\alpha}}$". We define an $(n+2)$-place function G_n^k on $\kappa_{n+1}^{+(n+2)}$:

$$G_n^k(\bar{\alpha}) = \{\langle r_i^{\bar{\alpha}}, \gamma_i^{\bar{\alpha}} \rangle : i < i(\bar{\alpha})\}.$$

The range of G_n^k has cardinality $\leqq \kappa_{n+1}$ (as $i(\alpha) < \kappa_n$ because $\Pi_{\ell \leqq n} P_\ell$ satisfies the κ_{n+1}-chain condition, and $r_i^{\bar{\alpha}} \in \Pi_{\ell \leqq n} P_\ell$, $|\Pi_{\ell \leqq n} P_\ell| = \kappa_{n+1}$; $\gamma_i^{\bar{\alpha}} < \kappa_n^{+(n+2)} < \kappa_{n+1}$ and $\kappa_{n+1}^{<\kappa_{n+1}} = \kappa_{n+1}$).

Let $B = \{t \in I_{n+1} : G_n^k$ has the same value on all increasing sequences of length $(n+2)$ from $t\}$. By definition

$$B \in D_{n+1}(\kappa_{n+1}^{+(n+2)}, \kappa_{n+1}^{+1}, \kappa_{n+1}) \subseteq D_{n+1}^*.$$

Hence $B' = \{\langle \alpha, t\rangle \in J_{n+1} : t \in B\} \in F_{n+1}$.

For every $\langle \alpha, t\rangle \in A^k_{n+1}$, choose an increasing sequence of length $(n+2)$ from $t, \bar{\beta}$, and we can find $q^{k+1}_{\langle \alpha,t\rangle}$, $q^k_{\langle \alpha,t\rangle} \leqq q^{k+1}_{\langle \alpha,t\rangle} \in P_n$, and $q^{k+1}_{\langle \alpha,t\rangle}$ force $\langle \bar{\gamma}, \underset{\sim}{f}{}^k_{n+1}(\bar{\gamma}\,\hat{}\,\bar{\beta}) : \bar{\gamma}$ an increasing finite sequence from $\kappa_n^{+(n+1)}\rangle$ to be equal to some $\Pi_{\ell<n} P_\ell$-name $\underset{\sim}{f}{}^k_{\langle \alpha,t\rangle}$ (possible as P_n is $\kappa_n^{+(n+2)}$-complete). If $\langle \alpha, t\rangle \in B'$ too, then the choice of $\bar{\beta}$ is immaterial. Now by Claim 9, we can find $A^{k+1}_{n+1} \subseteq B' \cap A^k_{n+1}$, as required, and as the number of possible $f^\alpha_{\langle \alpha,t\rangle}$ is $\leqq \kappa_n^{+(n+2)}$ we can assume $f^k_{\langle \alpha,t\rangle} = f^{k+1}_n$ for every $\langle \alpha, t\rangle \in A^k_{n+1}$.

This really finishes the proof.

We define $A^\omega_\ell = \bigcap_{k<\omega} A^k_\ell$, $q^\omega_{\langle \alpha,t\rangle} = \bigcup_{k<\omega} q^k_{\langle \alpha,t\rangle}$ and $p^\omega_\ell = \bigcup_{k<\omega} p^k_\ell$ for $\langle \alpha, t\rangle \in A^\omega_\ell$. As each F_ℓ is κ_ℓ-complete, $A^\omega_\ell \in F_\ell$. It is also clear that $p^\omega_\ell \in P_\ell$ and $q^\omega_{\langle \alpha,t\rangle} \in P_\ell$ for $\langle \alpha, t\rangle \in A^\omega_\ell$.

Choose $\langle \alpha_\ell, t_\ell\rangle \in A^\omega_\ell$, and let $p^1 = \langle q^\omega_{\langle \alpha_0,t_0\rangle}, q^\omega_{\langle \alpha_1,t_1\rangle}, \cdots, q^\omega_{\langle \alpha_\ell,t_\ell\rangle}, \cdots\rangle$ and $S_\ell = t_\ell$. It is easy to check they are as required.

CONCLUDING REMARKS. An alternative presentation of the proof is that, after the collapse, the filter that D^*_{n+1} generates (over $\mathrm{Inc}(\kappa_n^{+(n+1)}, \kappa_n^{+1})$) is still κ_{n+1}-complete, and it has the $\kappa_n^{+(n+1)}$-Laver property, i.e., there is a family S of subsets of I_{n+1} ($\in V$) which is $\kappa_n^{+(n+2)}$-complete (i.e., the intersection of any descending ω chain of members of S is in S (or just contain a member)), is dense (if $A \subseteq I_n$, $I_n - A \notin D^*_{n+1}$ then A contains a member of S), and $A \in \mathrm{S} \rightarrow A \subseteq I_n \wedge I_n - A \notin D^*_{n+1}$.

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